

Gauge fields with respect to $d = (3 + 1)$ in the Kaluza-Klein theories and in the *spin-charge-family theory*

Dragan Lukman, Norma Susana Mankoč Borštnik ^a

¹University of Ljubljana, FMF, Dept. of Physics, Jadranska 19, 1000 Ljubljana, Slovenia

Received: date / Accepted: date

Abstract It is shown that in the *spin-charge-family* theory [1,2,3,4,5], as well as in all the Kaluza-Klein like theories [6,7], vielbeins and spin connections manifest in $d = (3 + 1)$ space equivalent vector gauge fields, when space with $d \geq 5$ has large enough symmetry. The authors demonstrate this equivalence in spaces with the symmetry of the metric tensor in the space out of $d = (3 + 1)$ - $g^{\sigma\tau} = \eta^{\sigma\tau} f^2$ - for any scalar function f of the coordinates x^σ , where x^σ denotes coordinates of space out of $d = (3 + 1)$. Also the connection between vielbeins and scalar gauge fields in $d = (3 + 1)$ (offering the explanation for the Higgs's scalar) is discussed.

Keywords Unifying theories · Beyond the standard model · Properties of scalar fields · Origin and properties of gauge bosons · Kaluza-Klein-like theories

PACS 12.60.-i · 11.10.Kk · 04.50.-h · 12.10.-g · 11.30.-j · 14.80.-j · 11.15.Ex · 12.90.+b

1 Introduction

The *spin-charge-family* theory [1,2,3,4,5] explains, starting with the simple action (Eq. (1)) in $d > (3 + 1)$, all the assumptions of the *standard model*, as well as other phenomena, like the matter-antimatter asymmetry, dark matter appearance and others. In this theory the spin connection fields manifest in the low energy regime as the known vector gauge fields as well as the Higgs's scalars (and the Yukawa couplings), while in the Kaluza-Klein theories [6,7] vielbeins (or rather metric tensors) are usually used to represent vector gauge fields.

We demonstrate in this paper that in d -dimensional spaces with the symmetry of the metric tensor in $(d -$

4)-dimensional space $g_{\sigma\tau} = \eta_{\sigma\tau} f^{-2}$ [where (x^σ, x^τ) determine the coordinates of the almost compactified space [8,9,10], $\eta_{\sigma\tau}$ is the diagonal matrix in this space and f is any scalar function of these coordinates] both procedures - the ordinary Kaluza-Klein one with vielbeins and the one with spin connections (related to the vielbeins, Eq.(17)), used in the *spin-charge-family* theory ([1,2,3,4,5] and the references therein) - lead in $d = (3 + 1)$ to the same vector gauge fields. That either the vielbeins or the spin connections represent in symmetric enough $(d - 4)$ spaces in $d = (3 + 1)$ the same vector gauge fields is known for a long time [6,7,9].

This paper is to clarify the equivalence of representing in theories with higher dimensional spaces vector gauge fields either with spin-connections or with vielbeins, but it is also to show that expressing the gauge fields with spin connections rather than with vielbeins makes the *spin-charge-family theory* transparent and correspondingly elegant, so that it is easier to recognize that the origin of charges of the observed spinors, vector gauge fields, Higgs's scalar and Yukawa couplings might really be in $(d - 4)$ space, and that this explanation might show a possible next step beyond the *standard model*.

Let us remind the reader that the vector gauge fields, which carry the space index $m = (0, 1, 2, 3)$, as well as the spinor fields, both observed in $d = (3 + 1)$, have in the Kaluza-Klein theories and in the *spin-charge-family* theory all the charges defined by the symmetry in $(d - 4)$ -dimensional space, while the (observed) dynamics of these fields is defined in $(3 + 1)$ space ¹.

¹It is demonstrated on the special case in Ref. [8] that the observed charges of spinors and of vector gauge fields (this is true also for the charges of the scalar fields) originate in the lowest value of M^{st} , that is in S^{st} .

^ae-mail: norma.mankoc@fmf.uni-lj.si

We present also the relation between the vielbeins and the spin connection fields for the scalar gauge fields - for the same symmetry of d -dimensional spaces ($g_{\sigma\tau} = \eta_{\sigma\tau} f^{-2}$ in $(d-4)$ -dimensional space). While the vector gauge fields carry the space index $m = (0, 1, 2, 3)$, the scalar gauge fields carry the space index ($s \geq 5$). Scalar gauge fields, carrying the space index $s = (7, 8)$, manifest in $d = (3+1)$ as the Higgs's scalar of the *standard model*, carrying the weak and the hyper charges ($\pm\frac{1}{2}, \mp\frac{1}{2}$), respectively [2, 3, 4, 1]. Scalar gauge fields carry besides the properties defined by the space index (like there are the weak and hyper charges when the space index $s = (7, 8)$) also the charges defined by the superposition of S^{st} (superposition are determined by the symmetry of $(d-4)$ space).

There are spinor fields (and possibly also scalar gauge fields) which are responsible for curling $(d-4)$ space, forcing the space to manifest the required symmetry (Eqs. (5)-(8)). Consequently vielbeins and spin connections of $(d-4)$ space reflect this symmetry and correspondingly these spinors (or possibly as well scalar gauge fields) do not enter into the relation of Eq. (4) ².

Let us start with the action of the *spin-charge-family* theory [2, 5, 3, 4, 1]. In this simple action in an even dimensional space ($d = 2n, d > 5$) ³ fermions interact with the vielbeins f^α_a and the two kinds of the spin-connection fields - $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$ - the gauge fields of $S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ and $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$, respectively:

$$\mathcal{A} = \int d^d x E \frac{1}{2} (\bar{\psi} \gamma^a p_{0a} \psi) + h.c. + \int d^d x E (\alpha R + \tilde{\alpha} \tilde{R}), \quad (1)$$

here $p_{0a} = f^\alpha_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha_a\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$,

$$R = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\omega_{ab\alpha, \beta} - \omega_{ca\alpha} \omega^c_{b\beta})\} + h.c.,$$

²If there are additional spinors, which do strongly influence the relation among vielbeins and spin connections, the spin connections are not any longer uniquely determined by the vielbeins, as demonstrated in Eq. (4). Then the symmetry of $(d-4)$ space might change further. It can happen, like in Ref. [8, 9, 10], that some of spinors stay massless after the break and the others do not, or like at the electroweak break when the symmetry of $(d-4)$ space breaks so that the weak and hyper charges break, keeping the electromagnetic charge unbroken [2, 1, 5, 3, 4], while some scalars gain constant values (called in the *standard model* the nonzero vacuum expectation values) independent of $(3+1)$ space coordinates.

³In the *spin-charge-family* theory d is chosen to be $(13+1)$, what makes the simple starting action in d to manifest in $(3+1)$ in the low energy regime all the observed degrees of freedom, explaining all the assumptions of the *standard model* as well as other observed phenomena [2, 1, 5, 3, 4].

$$\tilde{R} = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\tilde{\omega}_{ab\alpha, \beta} - \tilde{\omega}_{ca\alpha} \tilde{\omega}^c_{b\beta})\} + h.c. \quad ^4.$$

The action introduces two kinds of the Clifford algebra objects, γ^a and $\tilde{\gamma}^a$,

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+. \quad (2)$$

f^α_a are vielbeins inverted to e^α_α , Latin letters (a, b, \dots) denote flat indices, Greek letters (α, β, \dots) are Einstein indices, (m, n, \dots) and (μ, ν, \dots) denote the corresponding indices in $(0, 1, 2, 3)$, (s, t, \dots) and (σ, τ, \dots) denote the corresponding indices in $d \geq 5$:

$$e^\alpha_\alpha f^\beta_a = \delta^\beta_\alpha, \quad e^\alpha_\alpha f^\alpha_b = \delta^\alpha_b, \quad (3)$$

$E = \det(e^\alpha_\alpha)$ ⁵. The action \mathcal{A} offers the explanation for the origin and all the properties of the observed fermions (of the family members and families), of the observed vector gauge fields, of the Higgs's scalar and of the Yukawa couplings, explaining the origin of the matter-antimatter asymmetry, the appearance of the dark matter and predicts new scalars and new family to be observed at the LHC ([2, 1] and the references therein).

The spin connection fields and the vielbeins are related fields and, if there are no spinor (fermion) sources present (both kinds of, the one of S^{ab} and the one of \tilde{S}^{ab}) the spin connection fields are expressible with the vielbeins. In Ref. [5] (Eq. (C9)) the expressions relating the spin connection fields of both kinds with the vielbeins and the spinor sources are presented.

We present below the relation among the $\omega_{ab\alpha}$ fields and the vielbeins ([9], Eq. (6.5), where the relation

$$e^\alpha_{, \alpha} + \omega^\alpha_{b\alpha} e^b_{\beta} - \Gamma^{\alpha'}_{\beta\alpha} e^\alpha_{\alpha'} = 0$$

is used, with $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\beta\delta, \gamma} + g_{\gamma\delta, \beta} - g_{\beta\gamma, \delta})$, ([3], Eq. (C9)).

$$\begin{aligned} \omega_{ab}{}^e &= \frac{1}{2E} \{e^\alpha_\alpha \partial_\beta (E f^\alpha_{[a} f^{\beta b]}) - e_{a\alpha} \partial_\beta (E f^\alpha_{[b} f^{\beta e]}) \\ &\quad - e_{b\alpha} \partial_\beta (E f^{\alpha[e} f^{\beta a]})\} \\ &\quad + \frac{1}{4} \{\bar{\Psi} (\gamma^e S_{ab} - \gamma_{[a} S_{b]}{}^e) \Psi\} \\ &\quad - \frac{1}{d-2} \left\{ \delta_a^e \left[\frac{1}{E} e^d_\alpha \partial_\beta (E f^\alpha_{[d} f^{\beta b]}) + \bar{\Psi} \gamma_d S^d{}_b \Psi \right] \right. \\ &\quad \left. - \delta_b^e \left[\frac{1}{E} e^d_\alpha \partial_\beta (E f^\alpha_{[d} f^{\beta a]}) + \bar{\Psi} \gamma_d S^d{}_a \Psi \right] \right\}. \end{aligned} \quad (4)$$

⁴Whenever two indexes are equal the summation over these two is meant.

⁵This definition of the vielbein and the inverted vielbein is general, no specification about the curled space is assumed yet. In Eq.(9) vielbeins are specified for the case that $(d-4)$ space is curled, Eqs. (15,16), while f^σ_m determines vector gauge fields Ω^{st}_m as presented in Eq. (14).

When the gauge vector and scalar fields in $d = (3 + 1)$ are studied, with the charges originating in $(d - 4)$ -dimensional space, the denominator $\frac{1}{d-2}$ must be replaced by $\frac{1}{(d-4)-2}$. One notices that if there are no spinor sources present, carrying the spinor quantum numbers S^{ab} , then ω_{abc} is completely determined by the vielbeins (and so is $\tilde{\omega}_{abc}$ if there are no spinor sources present carrying \tilde{S}^{ab}). Eq. (4) manifests that the last terms with δ_a^e and δ_b^e do not contribute when the vector gauge fields ω_{st}^m , $(s, t) = (5, 6, \dots, d)$ and $m = (0, 1, 2, 3)$, are under consideration.

We demonstrate in this paper, Sect. 2, that in the spaces with the maximal number of the Killing vectors ([6], p. (331–340)) and with no spinor sources present (which would change the symmetry of $(d-4)$ space), the vielbeins f^σ_m and the spin connections ω_{stm} are in the Kaluza-Klein theories [7, 6] related. We find, Eqs. (14, 18): $f^\sigma_m = -\frac{1}{2} E_{st}^\sigma \omega_{st}^m(x^\nu)$. When the vector gauge fields are superposition of the spin connection fields ($A_m^{Ai} = \sum_{s,t} c^{Ai}_{st} \omega_{st}^m$), the relations among the vielbeins and spin connections are correspondingly: $f^\sigma_m = \sum_A \tau^{A\sigma} \mathbf{A}_m^A$, as presented in Eqs. (20–23). Spinors, vector gauge fields and scalar gauge fields, manifested in $d = (3 + 1)$ as dynamical fields, can be treated as weak fields, which do not influence the symmetry of $(d-4)$ space. When these fields start to be strong the symmetry of the curled space changes. (Let us mention that, for example, scalar fields at the electroweak break do break the symmetry of $(d-4)$ space.)

Since the vielbeins f^α_a and inverted vielbeins e^a_α (Eq. (3)) appear in the metric tensor as a product ($g^{\alpha\beta} = f^\alpha_a f^{\beta a}$, $g_{\alpha\beta} = e^a_\alpha e_{a\beta}$), also tensors of the vector gauge fields appear in $d = (3 + 1)$ in the curvature $R^{(d)}$ as it is expected for the vector gauge fields, Eqs. (23, 24, 25):

$$R^{(d)} = R^{(4)} + R^{(d-4)} - \frac{1}{4} g_{\sigma\tau} E_{st}^\sigma E_{s't'}^\tau F_{\mu\nu}^{st} F^{s't'\mu\nu}.$$

We demonstrate in Sect. 3 that also spin connection fields $\omega_{st}^{s'}$ (with the index s' from $(d-4)$ space, and accordingly scalar with respect to $(3 + 1)$ space) are uniquely expressible by vielbeins, Eqs. (39, 40), as long as the curled space has large enough symmetry. Consequently also the superposition of the scalar spin connection fields are expressible with the vielbeins.

2 Proof that spin connections and vielbeins lead to the same vector gauge fields in $(3 + 1)$ -dimensional space-time

We discuss relations between spin connections and vielbeins when space in $(d-4)$ demonstrates the desired isometry in order to prove that both ways, either using

vielbeins or spin connections, lead to equivalent vector gauge fields in $(3 + 1)$.

We point out that spin connections manifest (charges and properties of) vector gauge fields more transparently (and elegantly) than vielbeins⁶.

Let $(d-4)$ space manifest the rotational symmetry, determined by the infinitesimal coordinate transformations of the kind

$$x'^\mu = x^\mu, \\ x'^\sigma = x^\sigma + \varepsilon^{st}(x^\mu) E_{st}^\sigma(x^\tau) = x^\sigma - i\varepsilon^{st}(x^\mu) M_{st} x^\sigma, \quad (5)$$

where $M^{st} = S^{st} + L^{st}$, $L^{st} = x^s p^t - x^t p^s$, S^{st} concern internal degrees of freedom of boson and fermion fields, $\{M^{st}, M^{s't'}\}_- = i(\eta^{st'} M^{ts'} + \eta^{ts'} M^{st'} - \eta^{ss'} M^{tt'} - \eta^{tt'} M^{ss'})$ ⁷. From Eq. (5) it follows

$$-i M_{st} x^\sigma = E_{st}^\sigma = x_s f^\sigma_t - x_t f^\sigma_s, \\ E_{st}^\sigma = (e_{s\tau} f^\sigma_t - e_{t\tau} f^\sigma_s) x^\tau, \\ M_{st}^\sigma := i E_{st}^\sigma, \quad (6)$$

and correspondingly: $M_{st} = E_{st}^\sigma p_\sigma$. One derives, when taking into account Eq. (6) and the commutation relations among generators of the infinitesimal rotations, the equation for the Killing vectors E_{st}^σ

$$E_{st}^\sigma p_\sigma E_{s't'}^\tau p_\tau - E_{s't'}^\sigma p_\sigma E_{st}^\tau p_\tau = \\ -i(\eta_{st'} E_{ts'}^\tau + \eta_{ts'} E_{st'}^\tau - \eta_{ss'} E_{tt'}^\tau - \eta_{tt'} E_{ss'}^\tau) p_\tau, \quad (7)$$

and the Killing equation

$$D_\sigma E_{\tau st} + D_\tau E_{\sigma st} = 0, \\ D_\sigma E_{\tau st} = \partial_\sigma E_{\tau st} - \Gamma^{\tau'}_{\sigma\tau} E_{\tau' st}. \quad (8)$$

Let the corresponding background field ($g_{\alpha\beta} = e^a_\alpha e_{a\beta}$) be

$$e^a_\alpha = \begin{pmatrix} \delta^m_\mu & e^m_\sigma \\ e^s_\mu & e^s_\sigma \end{pmatrix}, \quad f^\alpha_a = \begin{pmatrix} \delta^\mu_m & f^\sigma_m \\ 0 = f^\mu_s & f^\sigma_s \end{pmatrix}, \quad (9)$$

⁶ In addition: At low energies there are superposition of spins of spinors, which manifest charges of spinors in $(3 + 1)$, and there are superposition of S^{st} acting on superposition of spin connection fields which manifest as the charges of vector (and scalar) gauge fields (vectors manifest in addition to charges - originating $(d-4)$ - in $SO(3 + 1)$ the $SU(2) \times SU(2)$ spin structure, while scalars carry besides charges - originating $(d-4)$, of the same origin as there are charges of vector gauge fields - also the properties defined by the space index in $(d-4)$. All these support the idea that the origin of vector (as well as scalar) gauge fields might indeed be in higher dimensional space.

⁷ While L^{st} act on coordinates, S^{st} act on spinor fields, on vector gauge fields (they are superposition of ω_{stm} , (s, t) belonging to $(d-4)$ space, m to $(3 + 1)$ space) and on scalar gauge fields (they are superposition of $\omega_{stt'}$, (s, t, t') belonging to $(d-4)$ space), the charges of which originate in higher dimensional space and correspondingly S^{st} act on their charges (which are the superposition of S^{st}). For example, S^{ab} act on gauge fields [1] as follows: $S^{ab} A^{d\dots e\dots g} = i(\eta^{be} A^{d\dots a\dots g} - \eta^{ae} A^{d\dots b\dots g})$.

so that the background field in $d = (3 + 1)$ is flat. From $e^a{}_\mu f^\sigma{}_a = \delta^\sigma_\mu = 0$ it follows

$$e^s{}_\mu = -\delta^m_\mu e^s{}_\sigma f^\sigma{}_m. \quad (10)$$

This leads to

$$g_{\alpha\beta} = \begin{pmatrix} \eta_{mn} + f^\sigma{}_m f^\tau{}_n e^s{}_\sigma e_{s\tau} & -f^\tau{}_m e^s{}_\tau e_{s\sigma} \\ -f^\tau{}_n e^s{}_\tau e_{s\sigma} & e^s{}_\sigma e_{s\tau} \end{pmatrix}, \quad (11)$$

and

$$g^{\alpha\beta} = \begin{pmatrix} \eta^{mn} & f^\sigma{}_m \\ f^\sigma{}_s f^{\tau s} + f^\sigma{}_m f^{\tau m} & \end{pmatrix}. \quad (12)$$

We have: $\Gamma^{\tau'}{}_{\tau\sigma} = \frac{1}{2} g^{\tau'\sigma'} (g_{\sigma\sigma'}{}_{,\tau} + g_{\tau\sigma'}{}_{,\sigma} - g_{\sigma\tau}{}_{,\sigma'})$.

One can check properties of $f^\sigma{}_m \delta^\mu_m$ under general coordinate transformations: $x'^\mu = x'^\mu(x^\nu)$, $x'^\sigma = x'^\sigma(x^\tau)$, $(g'_{\alpha\beta} = \frac{\partial x^\rho}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} g_{\rho\delta})$,

$$f'^\sigma{}_m \delta^\mu_m = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial x'^\sigma}{\partial x^\tau} f^\tau{}_\nu. \quad (13)$$

Let us introduce the vector gauge field $\Omega^{st}{}_m(x^\nu)$, which depends only on the coordinates in $d = (3 + 1)$, as follows

$$f^\sigma{}_m = -\frac{1}{2} E^\sigma{}_{st} \Omega^{st}{}_m(x^\nu), \quad (14)$$

with $E^\sigma{}_{st} = -iM_{st}{}^\sigma$ defined in Eq. (6). $f^\sigma{}_m$ depends on the $(3+1)$ coordinates through $\Omega^{st}{}_m$ and on $(d - 4)$ coordinates through $E^\sigma{}_{st}$ ⁸. From Eqs. (13,14) the transformation properties of $\Omega^{st}{}_m$ under the coordinate transformations of Eq. (5) follow.

If we look for the transformation properties of the superposition of the fields Ω_{stm} , let say

$$\mathcal{A}^{Ai}{}_m = \sum_{s,t} c^{Aist} \Omega_{stm},$$

which are the gauge fields of τ^{Ai} (with the commutation relations $\{\tau^{Ai}, \tau^{Bj}\}_- = i\delta_B^A f^{Aijk} \tau^{Ak}$, where $\tau^{Ai} = \sum_{s,t} c^{Aist} M^{st}$ and f^{Aijk} are the structure constants of the corresponding gauge groups), under the coordinate transformations of Eq. (5), one finds $\delta_0 \mathcal{A}^{Ai}{}_m = \varepsilon^{Ai}{}_{,m} + i f^{Aijk} \mathcal{A}_m^{Aj} \varepsilon^{Ak}$.

Let us make a choice of $f^\sigma{}_s$

$$\begin{aligned} f^\sigma{}_s &= f \delta_s^\sigma, \\ e^s{}_\sigma &= f^{-1} \delta^s_\sigma, \end{aligned} \quad (15)$$

⁸These gauge fields $\Omega^{st}{}_m$ are in the low energy regime weak in comparison with the fields which force the space to curl. The influence of these gauge fields and correspondingly of $f^\sigma{}_m$ on the equations of motion of vielbeins and spin connections in the higher dimensional space can be assumed as negligible, as is the case for weak spinor sources (this is the usual procedure in problems in classical or quantum mechanics when weak perturbation is put into strong fields, this procedure is assumed also in the Kaluza-Klein theories).

for which $E^\sigma{}_{st}$ is equal to

$$E^\sigma{}_{st} = (\eta_{s\tau} \delta_t^\sigma - \eta_{t\tau} \delta_s^\sigma) x^\tau, \quad (16)$$

solving the Killing equation (8) if f is the scalar function of the coordinates. Let us put the expression for $f^\sigma{}_m$, Eq. (14), into Eq. (4) to see the relation among ω_{stm} and $f^\sigma{}_m$. One finds

$$\begin{aligned} \omega_{stm} &= \frac{1}{2E} \{ f^\sigma{}_m [e_{t\sigma} \partial_\tau (E f^\tau{}_s) - e_{s\sigma} \partial_\tau (E f^\tau{}_t)] \\ &\quad + e_{s\sigma} \partial_\tau [E (f^\sigma{}_m f^\tau{}_t - f^\tau{}_m f^\sigma{}_t)] \\ &\quad - e_{t\sigma} \partial_\tau [E (f^\sigma{}_m f^\tau{}_s - f^\tau{}_m f^\sigma{}_s)] \}. \end{aligned} \quad (17)$$

(Since we study only the relation between vielbeins and spin connections when there are no spinor sources present, either weak or strong, the term $\psi^\dagger \gamma^0 \gamma^m S_{st} \psi$ is dropped. Studying problems with the weak spinor sources present would only slightly complicate the problem, while it would make the proof less transparent.)

Using the inverse vielbeins $e^s{}_\sigma = f^{-1} \delta^s_\sigma$ and

$$\det(e^s{}_\sigma) = E = f^{-(d-4)}$$

(Eq. (9)) and taking $\Omega_{stm} = \Omega_{stm}(x^n)$, as assumed above, it follows (after using Eq. (14) and recognizing that $f^\sigma{}_m = -\frac{1}{2} (e_{s'\tau'} f^{\sigma\tau'} - e_{t'\tau'} f^{\sigma s'}) x^{\tau'} \Omega^{s't'}{}_m$)

$$\begin{aligned} \omega_{stm} &= \frac{1}{2} (\eta_{s\sigma} \delta_t^\sigma - \eta_{t\sigma} \delta_s^\sigma) \partial_\tau f_m^\sigma, \\ \omega_{stm} &= \Omega_{stm}. \end{aligned} \quad (18)$$

It is therefore proven for the vielbeins

$$f^\sigma{}_m = -\frac{1}{2} E^\sigma{}_{st} \omega^{st}{}_m(x^\nu),$$

Eq. (14), where in $d \geq 5$ vielbeins solve the Killing equation (8), that the spin connections determine the gauge vector fields in $d = (3 + 1)$.

Statement: Let the space with $s \geq 5$ have the symmetry allowing the infinitesimal transformations of the kind

$$x'^\mu = x^\mu, \quad x'^\sigma = x^\sigma - i \sum_{A,i,s,t} \varepsilon^{Ai}(x^\mu) c_{Ai}{}^{st} M_{st} x^\sigma, \quad (19)$$

then the vielbeins $f^\sigma{}_m$ in Eq. (9) manifest in $d = (3 + 1)$ the vector gauge fields \mathcal{A}_m^{Ai}

$$f^\sigma{}_m = \sum_A \tau^{A\sigma} \mathcal{A}_m^A, \quad (20)$$

where

$$\begin{aligned}
\tau^{Ai} &= \sum_{s,t} c^{Ai}_{st} M^{st}, \\
\{\tau^{Ai}, \tau^{Bj}\}_- &= i f^{Aijk} \tau^{Ak} \delta^{AB}, \\
\tau^A &= \tau^{A\sigma} p_\sigma = \tau^{A\sigma} \tau x^\tau p_\sigma \\
\tau^{Ai\sigma} &= \sum_{s,t} -i c^{Ai}_{st} M^{st\sigma} \\
&= \sum_{s,t} c^{Ai}_{st} (e_{s\tau} f^\sigma_t - e_{t\tau} f^\sigma_s) x^\tau = E^\sigma_{Ai}, \\
\mathcal{A}^{Ai}_m &= \sum_{s,t} c^{Ai}_{st} \omega^{st}_m. \tag{21}
\end{aligned}$$

The relation between ω^{st}_m and vielbeins is determined by Eq. (17).

We have to express $A^{Ai}_m = \sum_{s,t} c^{Aist} \omega_{stm}$ using Eq. (17). Then it is not difficult to see that we end up with the relation

$$A^{Ai}_m = \mathcal{A}^{Ai}_m, \tag{22}$$

leading to the equation

$$f^\sigma_m = \sum_A \tau^{A\sigma} \mathbf{A}_m. \tag{23}$$

The Lagrange function for these vector gauge fields follows from the curvature in d dimensional space

$$R = R^\alpha_{\beta\alpha\gamma} g^{\beta\gamma},$$

after using Eqs. (11,12) in the relation for $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\gamma\delta,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta})$ and after taking into account this relation in the Riemann tensor $R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta[\gamma,\delta]} + \Gamma^\alpha_{\alpha'[\gamma} \Gamma^{\alpha'}_{\beta\delta]}$, where $_{,\delta}$ denotes the derivative with respect to x^δ ($\frac{\partial}{\partial x^\delta}$) and the parentheses require antisymmetrization of the two indexes.

For a flat four dimensional space ($R^{(4)} = 0$) it follows for the curvature ([6], Eq. (10.41))

$$\begin{aligned}
R^{(d)} &= R^{(d-4)} - \frac{1}{4} g_{\sigma\tau} E^\sigma_{st} E^\tau_{s't'} F^{st}_{mn} F^{s't'}{}^{mn}, \\
F^{st}_{mn} &= \partial_m A^{st}_n - \partial_n A^{st}_m - f^{st}_{s't's''t''} A^{s't'}_m A^{s''t''}_n, \\
f^\sigma_m &= -\frac{1}{2} E^\sigma_{st} \omega^{st}_\mu f^\mu_m, \\
E^\sigma_{st} &= -i M^{st} x^\sigma = (e_{s\tau} f^\sigma_t - e_{t\tau} f^\sigma_s) x^\tau, \tag{24}
\end{aligned}$$

where $R^{(d-4)}$ determines the curvature in $(d-4)$ dimensional space and $f^{st}_{s't's''t''}$ can be obtained from the commutation relations $\{M^{st}, M^{s't'}\}_- = i(\eta^{st'} M^{ts'} + \eta^{ts'} M^{st'} - \eta^{ss'} M^{tt'} - \eta^{tt'} M^{ss'})$. Vielbein f^σ_m simplifies, when $f^\sigma_s = f^\sigma_s$ and $d = (3+1)$ is a flat space, to $f^\sigma_m = \omega^\sigma_{\tau m} x^\tau$.

When $(d-4)$ space manifests the symmetry of Eq. (20) ($f^\sigma_m = \sum_A \tau^{A\sigma} \mathbf{A}_m$) and $d = (3+1)$ is a flat space,

the curvature $R^{(d)}$ becomes equal to [6] (Eq. (10.41))⁹

$$\begin{aligned}
R^{(d)} &= R^{(d-4)} \\
&\quad - \frac{1}{4} \sum_{\substack{A,i,A',i', \\ \sigma,\tau,\mu,\nu}} g_{\sigma\tau} E^\sigma_{Ai} E^\tau_{A'i'} F^{Ai}_{mn} F^{A'i'}{}^{mn}, \\
F^{Ai}_{mn} &= \partial_m A^{Ai}_n - \partial_n A^{Ai}_m - i f^{Aijk} A^{Aj}_m A^{Ak}_n, \\
A^{Ai}_m &= \sum_{s,t} c^{Ai}_{st} \omega^{st}_m, \\
\tau^{Ai} &= \sum_{s,t} c^{Aist} M_{st}, \tag{25}
\end{aligned}$$

with E^τ_{Ai} defined in Eq. (21).

The integration of the action $\int E d^4x d^{(d-4)}x R^{(d)}$ over an even dimensional $(d-4)$ space leads to the well known effective action for the vector gauge fields in $d = (3+1)$ space: $\int E' d^4x \{-\frac{1}{4} \sum_{A,i,m,n} F^{Ai}_{mn} F^{Ai}{}^{mn}$, where E' is determined by the gravitational field in $(3+1)$ space ($E' = 1$, if $(3+1)$ space is flat). All the vector gauge fields (manifesting in $d = (3+1)$, x^m are coordinates in a flat $(3+1)$ space) are superposition of the spin connection fields: $A^{Ai}_m = \sum_{s,t} c^{Ai}_{st} \omega^{st}_m$, the charges of which ($\tau^{Ai} = \sum_{s,t} c^{Aist} S^{st}$) are determined by the symmetry of $(d-4)$ space.

This completes the proof of the above statement, that the vielbeins f^σ_m , $\sigma = (5, 6, \dots, d)$, $m = (0, 1, 2, 3)$, are expressible with the spin connection fields ω_{stm} : $f^\sigma_m = \sum_{A,i,s,t} \tau^{Aist} c_{Ai}{}^{st} \omega_{stm}$ ¹⁰.

Since vector gauge fields are direct ($c_{Ai}{}^{st}$ are complex numbers) superposition of spin connection fields, the spin connection fields offer an elegant and transparent description of the vector gauge fields. This is what the *spin-charge-family* theory is using.

In Subsect. 2.1 we demonstrate the connection among the spin connection fields ω_{stm} and the vielbeins f^σ_m when $(d-4)$ space manifests the $SU(2) \times SU(2)$ symmetry. Generalization to any symmetry in $(d-4)$ space goes in a similar way, leading to the corresponding expressions for the vector gauge fields in $d = (3+1)$.

2.1 Vector gauge fields $SU(2) \times SU(2)$ as the superposition of the spin connections

Let us demonstrate the statement that all the vector gauge fields are superposition of the spin connection fields in the case, when the space of the symmetry $SO(7,1)$ breaks into $SO(3,1) \times SU(2) \times SU(2)$.

⁹ Ref. [11], Sect. 5.3, deriving the Lagrange function for the gauge fields by using the Clifford algebra space, allows both, the curvature R and its quadratic form R^2 , Eq. (240).

¹⁰In general not only S^{st} but the total angular momentum $M^{st} (= L^{st} + S^{st})$ contribute to the charges of the vector gauge fields, manifesting in this case higher charges [8,9,10], but this is not what manifests in the low energy region.

One finds the coefficients c^{Ai}_{st} for the two $SU(2)$ generators, $\tau^{1i} = \sum_{s,t} c^{1i}_{st} M^{st}$ and $\tau^{2i} = \sum_{s,t} c^{2i}_{st} M^{st}$ by requiring the commutation relations $\{\tau^{Ai}, \tau^{Bj}\}_- = \delta^{AB} f^{Aijk} \tau^{Ak}$,

$$\begin{aligned}\tau^1 &= \frac{1}{2} (M^{58} - M^{67}, M^{57} + M^{68}, M^{56} - M^{78}) \\ \tau^2 &= \frac{1}{2} (M^{58} + M^{67}, M^{57} - M^{68}, M^{56} + M^{78}),\end{aligned}\quad (26)$$

while one finds coefficients c^{1i}_{st} and c^{2i}_{st} for the corresponding gauge fields

$$\begin{aligned}\mathbf{A}_a^1 &= \frac{1}{2} (\omega_{58a} - \omega_{67a}, \omega_{57a} + \omega_{68a}, \omega_{56a} - \omega_{78a}) \\ \mathbf{A}_a^2 &= \frac{1}{2} (\omega_{58a} + \omega_{67a}, \omega_{57a} - \omega_{68a}, \omega_{56a} + \omega_{78a}),\end{aligned}\quad (27)$$

from the relation

$$\sum_A \tau^A \mathbf{A}_m^A = \sum_{s,t} M^{st} \omega_{stm}. \quad (28)$$

Taking into account Eq. (6) one finds

$$\begin{aligned}\tau^1 &= \tau^{1\sigma} p_\sigma = \tau^{1\sigma} x^\tau p_\sigma, \\ \tau^2 &= \tau^{2\sigma} p_\sigma = \tau^{2\sigma} x^\tau p_\sigma, \\ \tau^{1\sigma} \tau_\tau &= \frac{1}{2} (e^5_\tau f^{\sigma 8} - e^8_\tau f^{\sigma 5} - e^6_\tau f^{\sigma 7} + e^7_\tau f^{\sigma 6}, \\ &\quad e^5_\tau f^{\sigma 7} - e^7_\tau f^{\sigma 5} + e^6_\tau f^{\sigma 8} - e^8_\tau f^{\sigma 6}, \\ &\quad e^5_\tau f^{\sigma 6} - e^6_\tau f^{\sigma 5} - e^7_\tau f^{\sigma 8} + e^8_\tau f^{\sigma 7}), \\ \tau^{2\sigma} \tau_\tau &= \frac{1}{2} (e^5_\tau f^{\sigma 8} - e^8_\tau f^{\sigma 5} + e^6_\tau f^{\sigma 7} - e^7_\tau f^{\sigma 6}, \\ &\quad e^5_\tau f^{\sigma 7} - e^7_\tau f^{\sigma 5} - e^6_\tau f^{\sigma 8} + e^8_\tau f^{\sigma 6}, \\ &\quad e^5_\tau f^{\sigma 6} - e^6_\tau f^{\sigma 5} + e^7_\tau f^{\sigma 8} - e^8_\tau f^{\sigma 7}).\end{aligned}\quad (29)$$

The expressions for f^σ_m are correspondingly

$$f^\sigma_m = (\tau^{1\sigma} \tau \mathbf{A}_m^1 + \tau^{2\sigma} \tau \mathbf{A}_m^2) x^\tau. \quad (30)$$

Expressing the two $SU(2)$ gauge fields, \mathbf{A}_m^1 and \mathbf{A}_m^2 , with ω_{stm} as it is required in Eq. (27), then using for each ω_{stm} the expression presented in Eq. (17), in which f^σ_m is replaced by the relation in Eq. (30), then taking for $f^\sigma_s = f \delta_s^\sigma$, where f is a scalar function of the coordinates x^σ , $\sigma = (5, 6, \dots, 8)$ (in this case $e^s_\mu = -\delta^m_\mu e^s_\sigma f^\sigma_m$, Eq. (10)), it follows after a longer but straightforward calculation that

$$\begin{aligned}\mathbf{A}_m^1 &= \mathcal{A}_m^1, \\ \mathbf{A}_m^2 &= \mathcal{A}_m^2.\end{aligned}\quad (31)$$

One obtains this result for any component of A_m^{1i} and A_m^{2i} , $i = 1, 2, 3$, separately.

It is not difficult to see that the gauge fields, which are superposition of ω_{stm} , $(s, t) = (5, 6, \dots, d)$, demonstrate in $d = (3 + 1)$ the isometry of the space of

$SO(d - 4)$, Eq. (9), with

$$e^s_\sigma = f^{-1} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & 0 \\ & & & & \dots & 0 \\ 0 & 0 & \dots & & & 1 \end{pmatrix}, \quad (32)$$

$$f^\sigma_s = f \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & 0 \\ & & & & \dots & 0 \\ 0 & 0 & \dots & & & 1 \end{pmatrix}.$$

The space breaks into $SO(3 + 1) \times SO(d - 4)$ and f is any scalar field of the coordinates:

$$f = f\left(\frac{\sum_\sigma (x^\sigma)^2}{\rho_0^2}\right), \quad (33)$$

while ρ_0 is the radius of the $(d - 4)$ sphere and

$$f^\sigma_m = \sum_A \mathbf{A}_m^A \tau^{A\sigma} x^\tau, \quad (34)$$

where \mathbf{A}_m^A are the superposition of ω^{st}_m ,

$$A_m^{Ai} = \sum_{s,t} c^{Ai}_{st} \omega^{st}_m,$$

demonstrating the symmetry of space with $s \geq 5$. This illustrates the proof of the statement in the section 2.

3 Relations between vielbeins and spin connections for scalars

The *spin-charge-family* theory offers the explanation for the origin of the Higgs's scalar and the Yukawa couplings: The scalar gauge fields - the gauge fields of the charges described by the two kinds of the Clifford algebra objects $[2, 1]$, γ^a 's and $\tilde{\gamma}^a$'s, (Eq. (1)) - take care of masses of spinors after the electroweak break.

We discuss here only the relation between vielbeins and spin connections for scalars the charges of which have the same origin as the charges of the vector gauge fields and only as long as $(d - 4)$ space manifests the isometry presented in Eqs. (5-9) with the choice of $f^\sigma_s = f \delta_s^\sigma$, Eq. (15), (which solves the Killing equation (8), if f is the scalar function of the coordinates x^σ). We do not include fermion sources which would change the symmetry of $(d - 4)$ space, while f^σ_m are (in the low energy regime) weak fields. This section is only to point out the differences between the relation of spin connection - vielbeins for vector and scalar gauge fields.

Let us add that while the spin of the vector gauge fields in $(3+1)$ determines with respect to the space index $m = (0, 1, 2, 3)$ the $SU(2) \times SU(2)$ structure of their spin, the space index s of the superposition of the scalar spin connection fields - $\sum_{t,t'} c^{Aitt'} \omega_{tt's}$ - manifests for $s = (7, 8)$ the weak and hyper charges of the Higgs's scalar: $(\pm\frac{1}{2}, \mp\frac{1}{2})$, respectively. Superposition of the spin connection fields with the space indices > 8 take care of transitions from the matter to antimatter and back, contributing to the matter-antimatter asymmetry of our universe.

To find the relation between vielbeins and spin connections we need to express the curvature $R^\sigma_{\tau\sigma\tau'} g^{\tau\tau'}$ for $(d-4)$ space, where the Riemann tensor and $\Gamma^\sigma_{\tau\sigma'}$ for this space are

$$R^\sigma_{\tau\sigma'\tau'} = \Gamma^\sigma_{\tau[\tau',\sigma']} + \Gamma^\sigma_{\tau''[\sigma']} \Gamma^{\tau''}_{\tau\tau'}, \quad (35)$$

$$\Gamma^\sigma_{\tau\sigma'} = \frac{1}{2} g^{\sigma\tau'} (g_{\sigma'\tau',\tau} + g_{\tau\tau',\sigma'} - g_{\tau\sigma',\tau'}),$$

in terms of vielbeins $g^{\sigma\tau} = f^\sigma_s f^{\tau s}$, which is in our case $g^{\sigma\tau} = f^2 \eta^{\sigma\tau}$, while $g_{\sigma\tau} = f^{-2} \eta_{\sigma\tau}$ ($_{,\delta}$ again denotes the derivative with respect to x^δ and $[\]$ the antisymmetrization with respect to particular two indexes) and compare this expression with the corresponding one when R is expressed with spin connections (and with the vielbeins).

$$R = \frac{1}{2} \{ f^{\alpha[a} f^{\beta b]} (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \omega^c_{b\beta}) \} + h.c. \quad (36)$$

One finds that $\Gamma^\sigma_{\tau\sigma'}$ is for $f^\sigma_s = f \delta^\sigma_s$ equal to $\Gamma^\sigma_{\tau\sigma'} = f^{-1} (\delta^\sigma_{\sigma'} f_{,\tau} + \delta^\sigma_\tau f_{,\sigma'} - \eta_{\sigma'\tau} f^{,\sigma})$, while one finds for $\omega^{st}_{s'} = -(f^{,t} \delta^s_{s'} - f^{,s} \delta^t_{s'})$ and for $\omega^{st}_\sigma = \omega^{st}_{s'} e^{s'}_\sigma = -f^{-1} (f^{,t} \delta^s_\sigma - f^{,s} \delta^t_\sigma)$.

It then follows for $R = R^\sigma_{\tau\sigma\tau'} g^{\tau\tau'}$, Eq. (35), since $\Gamma^\sigma_{\tau[\tau',\sigma]} g^{\tau\tau'} = 2(d-4-1)(f_{,\tau} f^{,\tau'} - f f_{,\tau}{}^{,\tau'})$ and $\Gamma^\sigma_{\tau''[\sigma]} \Gamma^{\tau''}_{\tau\tau'} g^{\tau\tau'} = (-1+d-4)(2-d-4)f_{,\tau} f^{,\tau'}$, that

$$R = R^\sigma_{\tau\sigma\tau'} g^{\tau\tau'} = (d-4-1) \{ [2-(d-4-2)] \cdot f_{,\tau} f^{,\tau'} - 2 \cdot f f_{,\tau}{}^{,\tau'} \}. \quad (37)$$

We take into account Eq. (4) and evaluate Eq. (36), obtaining

$$f^{\sigma[s} f^{\tau t]} \omega_{st\sigma,\tau} = 2(d-4-1)(f_{,\tau} f^{,\tau} - f f_{,\tau}{}^{,\tau}),$$

$$f^{\sigma[s} f^{\tau t]} (-)\omega_{t's\sigma} \omega^{t'}_{t\tau} = (-1+d-4)(2-d-4)f_{,\tau} f^{,\tau},$$

what leads to

$$\frac{1}{2} \{ f^{\sigma[s} f^{\tau t]} (\omega_{st\sigma,\tau} - \omega_{t's\sigma} \omega^{t'}_{t\tau}) \} + h.c. =$$

$$(d-4-1) \{ [2-(d-4-2)] \cdot f_{,\tau} f^{,\tau} - 2 \cdot f f_{,\tau}{}^{,\tau} \}. \quad (38)$$

We conclude: If $f^\sigma_s = \delta^\sigma_s f$, where $f = f(x^\tau x_\tau)$, then both expressions for the curvature of $(d-4)$ space

- the one with the metric tensor (Eq. 35) and the one with the spin connection (Eq. 36) - lead, as expected to the same expression

$$R = R^\sigma_{\tau\sigma\tau'} g^{\tau\tau'} = \frac{1}{2} \{ f^{\sigma[s} f^{\tau t]} (\omega^{st}_{\tau,\sigma} + \omega_{st'\sigma} \omega^{t'}_{t\tau}) \} + h.c., \quad (39)$$

where

$$\omega^{st}_\sigma = \omega^{st}_{s'} e^{s'}_\sigma = -f^{-1} (f^{,t} \delta^s_\sigma - f^{,s} \delta^t_\sigma). \quad (40)$$

The result is valid also for the case that vielbeins and spin connections depend on the coordinates of $(3+1)$ space: $f = f(\rho, x^m)$, $\omega_{stt'} = \omega_{stt'}(x^\sigma, x^m)$, $m = (0, 1, 2, 3)$, $(s, t, t') = (5, 6, \dots, d)$.

That spin connections and vielbeins lead to the same Lagrange density in $(d-4)$ space, although as expected, contributes to better understanding how in the low energy regime, after the electroweak break, scalar fields expressed with spin connections $\omega_{stt'}$, $t' = (7, 8)$, offer the explanation for Higgs's scalar and the Yukawa couplings [1, 3].

4 Conclusions

In the Kaluza-Klein theories the vector gauge fields - the gauge fields of the charges originating in higher dimensional spaces - are represented through the vielbeins f^σ_m (Eq. (9)) or rather with the corresponding metric tensors (Eqs. (11,12)). In the *spin-charge-family* theory the vector gauge fields are expressed as superposition of the spin connection fields $A_m^{Ai} = \sum_{t,t'} c^{Aitt'} \omega_{tt'm}$. This presentation offers an elegant and transparent understanding of the appearance of the vector gauge fields A_m^{Ai} , the charges of which originate in this theory (and in the Kaluza-Klein theories) in higher dimensional spaces, while dynamics is determined in $(3+1)$.

Also the scalar (gauge) fields of the *spin-charge-family* theory originate in higher dimensional spaces, offering the explanation for the origin of the Higgs's scalar and Yukawa couplings of the *standard model* - when the scalar gauge fields of both charges, S^{st} and \tilde{S}^{st} (Eq. (2)), are taken into account [1]. Their dynamics is (like in the case of the vector gauge fields) determined in $(3+1)$. We discuss in this paper only gauge fields of S^{st} for either vector or scalar fields.

We presented the proof, that the vielbeins f^σ_m (Einstein index $\sigma \geq 5$, $m = 0, 1, 2, 3$) lead in $d = (3+1)$ to the vector gauge fields, which are the superposition of the spin connection fields ω_{stm} : $f^\sigma_m = \sum_A A_m^A \tau^{A\sigma}_\tau x^\tau$, with $A_m^{Ai} = \sum_{s,t} c^{Ai}_{st} \omega^{st}_m$, when the metric in $(d-4)$, $g_{\sigma\tau}$, is invariant under the coordinate transformations $x^{\sigma'} = x^\sigma + \sum_{A,i,s,t} \epsilon^{Ai}(x^m) c^{Ai}_{st} E^{\sigma st}(x^\tau)$ and $\sum_{s,t} c^{Ai}_{st} E^{\sigma st} = \tau^{A i \sigma}$, while $\tau^{A i \sigma}$ solves the Killing

equation (8): $D_\sigma \tau_\tau^{Ai} + D_\tau \tau_\sigma^{Ai} = 0$ ($D_\sigma \tau_\tau^{Ai} = \partial_\sigma \tau_\tau^{Ai} - \Gamma_{\tau\sigma}^{\tau'} \tau_\tau^{Ai}$).

We demonstrated for the case when $SO(7, 1)$ breaks into $SO(3, 1) \times SU(2) \times SU(2)$ that $\sum_{A,i} \tau_\tau^{Ai} A_m^{Ai} = \sum_{s,t} S^{st} \omega_{stm}$ and that the effective action in flat (3+1) space for the vector gauge fields is $\int d^4x \{ -\frac{1}{4} F_{mn}^{Ai} F^{Aimn} \}$, where $F_{mn}^{Ai} = \partial_m A_n^{Ai} - \partial_n A_m^{Ai} - i f^{Aijk} A_m^{Aj} A_n^{Ak}$, and f^{Aijk} are the structure constants of the corresponding gauge groups.

The generalization of the break of $SO(13, 1)$ into $SO(3, 1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$, used in the *spin-charge-family* theory, goes equivalently. In a general case one has $\sum_{A,i} \tau_\tau^{Ai} A_m^{Ai} = \sum_{s,t}^* S^{st} \omega_{stm}$, where $*$ means that the summation concerns only those (s, t) , which appear in $\tau_\tau^{Ai} = \sum_{s,t} c^{Aist} S^{st}$. These vector gauge fields A_m^{Ai} , expressible with the spin connection fields, $A_m^{Ai} = \sum_{s,t} c^{Aist} \omega_{stm}$, offer an elegant explanation for the appearance of the vector gauge fields in the observed (3+1) space. The proof is true for any f which is a scalar function of the coordinates $x^\sigma, \sigma \geq 5$.

We demonstrated also the relation between the spin connection fields and vielbeins for the scalar fields. While for the vector gauge fields the effective low energy action is in $d = (3+1)$ equal to $\int E' d^4x \{ -\frac{1}{4} F_{mn}^{Ai} F^{Aimn} \}$ - where $F_{mn}^{Ai} = \partial_m A_n^{Ai} - \partial_n A_m^{Ai} - i f^{Aijk} A_m^{Aj} A_n^{Ak}$, $A_m^{Ai} = \sum_{s,t} c^{Aist} \omega_{stm}$, $E' = 1$ in flat (3+1) space, $\tau_\tau^{Ai} = \sum_{s,t} c^{Aist} S^{st}$ and f^{Aijk} are structure constants of the corresponding gauge groups - it follows for the scalar fields that, Eqs. (35,36),

$$R = \{ \Gamma_{\tau[\tau', \sigma]}^\sigma + \Gamma_{\tau''[\sigma}^\sigma \Gamma_{\tau\tau']}^{\tau''} \} g^{\tau\tau'} \\ = \frac{1}{2} \{ f^{\sigma[s} f^{\tau t]} (\omega_{\tau, \sigma}^{st} + \omega_{st' \sigma} \omega_{t\tau}') \} + h.c. .$$

(The corresponding action is proportional to $\int E d^{d-4}x R$). Similar relation follows also for the superposition of the spin connection fields.

If $\omega_{st' \sigma}$ depend on x^m (x^m are coordinates in (3+1) space), the scalar fields are the dynamical fields in (3+1), explaining, for example, after the break of the starting symmetry, the appearance of the Higgs's scalars and the Yukawa couplings [1,2,3,4,5].

All these relations are valid as long as spinors and vector gauge fields are weak fields in comparison with the fields which force $(d-4)$ space to be curled. When all these fields, with the scalar gauge fields included, start to be comparable with the fields (spinors or scalars), which determine the symmetry of $(d-4)$ space, the symmetry of the whole space changes.

Appendix A: Derivation of the equality

$$A_m^1 = \mathcal{A}_m^1$$

We demonstrate for the case $A_m^{11} = (\omega_{58m} - \omega_{67m})$, Eq. (27), that this A_m^{11} is equal to \mathcal{A}_m^{11} , appearing in Eq. (30)

$$f^\sigma_m = \sum_{A,i} \mathcal{A}_m^{Ai} \tau_\tau^{Ais} x^\tau. \quad (A.1)$$

When using Eq. (17) for $A_m^{11} = \omega_{58m} - \omega_{67m}$ we end up with the expression

$$A_m^{11} = \frac{1}{2E} \left\{ f^\sigma_m [e^8_\sigma \partial_\tau (E f^{\tau 5}) - e^5_\sigma \partial_\tau (E f^{\tau 8})] \right. \\ - f^\sigma_m [e^7_\sigma \partial_\tau (E f^{\tau 6}) - e^6_\sigma \partial_\tau (E f^{\tau 7})] \\ + e^5_\sigma \partial_\tau [E (f^\sigma_m f^{\tau 8} - f^\tau_m f^{\sigma 8})] \\ - e^6_\sigma \partial_\tau [E (f^\sigma_m f^{\tau 7} - f^\tau_m f^{\sigma 7})] \\ - e^8_\sigma \partial_\tau [E (f^\sigma_m f^{\tau 5} - f^\tau_m f^{\sigma 5})] \\ \left. + e^7_\sigma \partial_\tau [E (f^\sigma_m f^{\tau 6} - f^\tau_m f^{\sigma 6})] \right\}. \quad (A.2)$$

Inserting for f^σ_m the expression from Eq. (A.1) we obtain, when taking into account Eq. (29),

$$A_m^{11} = \partial_8(f_{5m}) - \partial_5(f_{8m}) - \partial_7(f_{6m}) + \partial_6(f_{7m}). \quad (A.3)$$

Inserting Eq. (A.1), in which we take into account Eq. (29) as well as that $e^s_\sigma = f^{-1} \delta_\sigma^s$ and $f^\sigma_s = f \delta_s^\sigma$, into Eq. (A.3), we end up with

$$A_m^{11} = \sum_{A,i} \mathcal{A}_m^{Ai} \delta_1^A \delta_1^i. \quad (A.4)$$

Similarly one obtains for the gauge fields of both subgroups $SU(2) \times SU(2)$ of the group $SO(4)$

$$A_m^{Ai} = \sum_{B,j} \mathcal{A}_m^{Bj} \delta_B^A \delta_j^i. \quad (A.5)$$

Similar derivations go for any $SO(n)$.

References

1. N.S. Mankoč Borštnik, sent for publication into Proceedings to The 10th Biennial Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields, IARD conference, Ljubljana 6-9 of June 2016 [arXiv:1607.01618v2].
2. N.S. Mankoč Borštnik, Phys. Rev. **D 91**, 065004 (2015) [arXiv:1409.7791].
3. N.S. Mankoč Borštnik, J. of Mod. Physics **6** 2244 (2015), doi: 10.4236/jmp.2015.615230 [arXiv: 1409.4981].
4. N.S. Mankoč Borštnik, J. of Mod. Physics. **4**, 823 (2013) doi: 10.4236/jmp.2013.46113 [arXiv:1312.1542].
5. N.S. Mankoč Borštnik, "The *spin-charge-family* theory is offering an explanation for the origin of the Higgs's scalar and for the Yukawa couplings" [arXiv:1409.4981].
6. M. Blagojević, *Gravitation and gauge symmetries*, (IoP Publishing, Bristol, 2002)

-
7. The authors of the works presented in *An introduction to Kaluza-Klein theories*, ed. by H. C. Lee, (World Scientific, Singapore, 1983).
 8. D. Lukman, N.S. Mankoč Borštnik, H.B. Nielsen, *New J. Phys.* **13** (2011) 103027, 1 [arXiv:1001.4679v5].
 9. N. Mankoč Borštnik, H.B. Nielsen, D. Lukman, in *Proceedings to the 7th Workshop "What Comes Beyond the Standard Models"*, Bled, July 19 - 31, 2004, ed. by N. Mankoč Borštnik, H.B. Nielsen, C. Froggatt, D. Lukman (DMFA Založništvo, Ljubljana, December 2004), p. 64 [hep-ph/0412208].
 10. N.S. Mankoč Borštnik, H.B. Nielsen, *Phys. Lett. B* **633** (2006) 771 [arXiv:hep-th/0509101]
 11. M. Pavšič, *Int. J. Mod.Phys. A* **21**, 5905 (2006) [arXiv:gr-qc/0507053]